Three Point Boundary Value Problems Associated with First Order Fuzzy Difference Systems-Existence and Uniqueness via the Best Least Square Solution

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Abstract:
This paper presents a criteria for the existence and uniqueness of solutions to first order fuzzy difference system using QR-algorithm. Modified QR-algorithm is presented for fuzzy linear systems using singular value decomposition.

Keywords: Fuzzy Difference Systems, Modified QR-algorithm, Fundamental matrix, Decode algorithm.

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1. Introduction:
Existence and uniqueness of solutions to initial value problems have a long mathematical history going back to Picard’s. The mere fact that f is continuous on R ensures existence of at least one solution to the initial value problem

\[ y'(t,y), \quad y(t_0) = y_0 \]  \tag{1.1}

on R. The situation is different for boundary value problems. Length of interval estimates are necessary to prove existence and uniqueness of (1.1). If f satisfies a lipschitz condition in the second variable, then (1.1) has a unique solution. The situation is different for first – order difference system.

\[ y_{n+1} = A(n)y_n + f_n, \quad y(n_0) = y_0, \]  \tag{1.2}

where A is an p x p continuous matrix, whose elements \( a_{ij}(n) \) are all real or complex valued functions defined on \( N_{n_0}^+ \) and \( y_n \in \mathbb{R}^p(C^p) \) with components \( y_1(n), y_2(n), \ldots, y_p(n) \) defined on \( N_{n_0}^+ \). The corresponding homogeneous equation corresponding to (1.2) is

\[ y_{n+1} = A(n)y_n, \quad y(n_0) = y_0 \]  \tag{1.3}

(1.3) possess a unique solution on \( N_{n_0}^+ \) as can easily be seen by induction.
This paper presents a criteria for the existence and uniquness of solutions to the three point boundary values problems associated with first order matrix difference systems.

\[ y_{n+1} = A(n)y_n + f_n \]  \hspace{1cm} (1.4)

\[ My_{n_0} + Ny_{n_m} + Ry(n_f) = \alpha, \]  \hspace{1cm} (1.5)

where M, N and R constant matrices of order \((m \times p)\) and \(y\) is a \((p \times 1)\) vector and \(\alpha\) is a constant \((p \times 1)\) vector. The corresponding homogeneous boundary value problems

\[ y_{n+1} = A(n)y_n \]  \hspace{1cm} (1.6)

\[ My_{n_0} + Ny_{n_m} + Ry(n_f) = 0. \]  \hspace{1cm} (1.7)

Throughout this paper we assume that \(e_1, e_2, \ldots, e_d\) be standard base vectors in \(R^d\) and \(y(n, n_0, e_i)\) \(i=1,2,\ldots,d\) be a linearly independent solutions having \(e_i(i = 1,2,\ldots,d)\) as standard base vectors. Let \(S\) be the solution space of (1.5). It may be noted that any element of \(S\) can be expressed as a linear combination of the set of \(n\) linearly independent solutions of \(y(n, n_0, e_i), i = 1,2,\ldots,p\) if \(Z(n)\) is any solution of (1.5) then

\[ Z(n) = \sum_{i=1}^{p} c_i y(n, n_0, e_i) \]  \hspace{1cm} \(i = 1,2,\ldots,p\)

We define Wronskian of functions \(y_i(n), n = 1,2,\ldots,p\) on \(N_{n_0}^+\) as

\[ W(n) = \begin{bmatrix}
    y_1(n) & y_2(n) & \cdots & y_p(n) \\
    y_1(n+1) & y_2(n+1) & \cdots & y_p(n+1) \\
    \vdots & \vdots & \ddots & \vdots \\
    y_1(n+p-1) & y_2(n+p-1) & \cdots & y_p(n+p-1)
\end{bmatrix} \]

Note that \(|W(n)| \neq 0\) for all \(n \in N_{n_0}^+\). If \(y_i(n), i = 1,2,\ldots,p\) be \(p\) linearly independent solution of (1.5), then \(|W(n)| \neq 0\) for all \(n \neq n_0\) if \(y(n)\) is any solution of (1.4) and \(\bar{y}(n)\) is a particular solution of (1.4) then \(y(n) - \bar{y}(n)\) is a solutions of (1.5) and any solution \(y(n)\) of (1.1) is given by

\[ y(n) = \bar{y}(n) + \sum_{i=1}^{p} \alpha_i y(n, n_0, e_i) \]


\[ y^a(n+1) = A(n)y^a_n + f(n), \]  \hspace{1cm} (1.8)

satisfying boundary conditions
\[ My^\alpha (n_0) + Ny^\alpha (n_m) + Ry^\alpha (n_f) = \alpha. \quad (1.9) \]

By using modified QR-algorithm we develop QR-algorithm for fuzzy linear systems. Section 2 presents preliminary results on fuzzy differential discrete systems and establishes main result by using modified QR-algorithm for fuzzy linear systems. These are results in fact generalize all existing results on linear systems and includes them as a particular case. The algorithm we present is a centrally crucial problems and is helpful in solving many least square problems in numerical linear algebra.

2) Preliminaries:

We present in this section some of the basic results and definitions on fuzzy systems. The family of all non-empty compact convex subsets of \( R^d \) is denoted by \( P_k(R^d) \). If \( \alpha, \beta \in R \) and \( A, B \in P_k(R^d) \), we define

\[ \alpha(A + B) = \alpha A + \alpha B, \quad (\alpha \beta)A = (\alpha \beta)A \text{ and } A = A \]

If \( \alpha, \beta > 0 \), then \( (\alpha + \beta)A = \alpha A + \beta A \). Let \( T = [a,b] \) be a compact subinterval of \( R \). We have the following :

Definition 2.1:

Let \( E^n = \{ u \in R^d \rightarrow [0,1] \} \), \( u \in E^n \) is called a fuzzy number, if it satisfies the following axioms

i) \( u \) is normal , that is there exists an \( x_0 \in R^d \) such that \( u(x_0) = 1 \)

ii) \( u \) is fuzzy convex , that is for any \( x, y \in R^d \) and \( 0 < \lambda < 1, u(\lambda x + (1 - \lambda)y) \in R^d \)

iii) \( u \) is upper semi continuous

iv) \( u^0 = cl\{ x \in R^d / u(x) \geq 0 \} \) is compact .

For \( \alpha \in [0,1] \) the \( \alpha \) – level set \( \{ u \}^\alpha \in P_k(R^d) \).

Definition 2.1(fuzzy set). Let \( X \) be a non-empty set. A fuzzy set \( A \) in \( X \) characterized by its membership function \( A: X \rightarrow [0,1] \) and \( A(x) \) is interpreted as the degree of membership of elements of \( x \) in every fuzzy set \( A \) to each \( x \in X \)

The value of zero is used to represent complete non-membership, the value of one is used to represent complete membership and the values \( \in \) between 0 and 1 are used to represent intermediate degrees of membership.

Example 2.1: The membership function of the fuzzy set of real numbers close to one is defined as \( A(x) = e^{-\beta(x-1)^2}, \text{where } \beta > 0 \)

Example 2.2: Let the membership functions for the set of real real numbers aloes to zero is defined as \( B(x) = \frac{1}{1+x^3} \)

Using this function , we can determine the membership grade of real number in the fuzzy set , which signifies the degree to which that membership is close to zero. For instance the number 1 a grade of 0.5 and the number zero is a grade of 1. Mostly the results available in literature are of zero number and is of grade one only.

Definition 2.2: A map \( f: [0,1] \rightarrow E^d \) is strongly measurable if for all \( \alpha \in [0,1] \) the multivalued map \( f_\alpha: [0,1] \rightarrow P_k(R^d) \) is defined as \( f_\alpha(t) = [f(t)]^\alpha \)
is Lebesgue measurable, when \( P_k(R^d) \) is endowed with the topology by the Hausdorff metric \( d \).

**Theorem 2.1** if \( u \in P_k(R^d) \), then

1. \( [u]^{\alpha} \in P_k(N_{\infty}^+ \alpha) \) for all \( \alpha \in [0,1] \)
2. If \( [u]^{\alpha 2} < [u]^{\alpha 1} \) for all \( 0 \leq \alpha 1 \leq \alpha 2 \leq 1 \)
3. If \( \{\alpha_k\} \) is a non decreasing sequence converging to \( \alpha > 0 \), then \( [u]^{\alpha} = n[u]^{\alpha k} \)

Conversely, if \( A^\alpha: 0 \leq \alpha \leq 1 \) and \( [u]^0 = u_0 \) and \( A^\alpha \subset A^0 \)

**Definition 2.3:** we define \( D: E^d \times E^d \rightarrow R_+ u\{0\} \) by \( D(u,v) = sup_{0 \leq \alpha \leq 1} d_H([u]^{\alpha}, [v]^{\alpha}) \) where \( d_H \) is the Hausdorff metric defined in \( P_k(R^d) \) For any \( u,v,w \in P_k(R^d) \) and \( \lambda \in R \), we have

1. \( D(u + w, v + w) = D(u, v) \)
2. \( D(\lambda u, \lambda v) = |\lambda| D(u, v) \) and
3. \( D(u, v) = D(u, w) + D(w, v) \)

**Definition 2.4:** Let \( f:T \rightarrow E^d \) for \( t_0 \in R \), we say that \( f \) is differentiable at \( t_0 \) (Hausdorff differentiable) if there exists an element \( f'(t_0)e \in R^d \) such that for all \( h \geq 0 \) the H-difference \( f(t_0 + h) - f(t_0) \) and \( F(t_0) - F(t_0 - h) \) exists and the limit (in the metric)

\[
\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) - F(t_0)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{F(t_0) - F(t_0 - h)}{h}
\]

are all exists and each equal to \( f'(t_0) \). At the end points we only take one sided derivative.

We now turn our attention to the existence and uniqueness of three point boundary value problems when the characteristic matrix \( D \) is non-invertible. Let \( y^\alpha(n_0, n_0, e_i), i = 1, 2, \ldots, \alpha \) be the \( n \)-linearly independent solutions of (1.7) having \( e_i \) as its initial base vector. Then any solution of (1.7) is of the form

\[
y(n) = \sum_{n=n_0}^{d-1} \varphi^{-\alpha}(n, j + 1) f_j + \varphi^{-\alpha}(n, n_0) C \tag{2.1}
\]

where \( c \) is an constant \( n \) vector. Now substituting the general form of solution (2.1) in the boundary condition matrix(1.8)

We get

\[
[M \varphi^{-\alpha}(n, n_0) + N \varphi^{-\alpha}(n_m, n_0) + R \varphi^{-\alpha}(n_f, n_0) C] = \alpha - [\Sigma M \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_0, j + 1) f_j + N \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_m, j + 1) f_j + R \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_f, j + 1) f_j].
\]

We assume that for each \( \alpha \in [0,1] \), the characteristic matrix \( D^\alpha \) defined by

\[
D^\alpha = M \varphi^{-\alpha}(n, n_0) + N \varphi^{-\alpha}(n_m, n_0) + R \varphi^{-\alpha}(n_f, n_0)
\]

is non-singular (here we assume that \( M, N \) and \( R \) are constant square matrices). In a way we are assuming that the homogeneous boundary value problem (with \( f = 0 \) and \( \alpha = 0 \)) has only the trivial solution. In this case

\[
C_{n_0} = D^{-1}[\alpha - [\Sigma M \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_0, j + 1) f_j + N \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_m, j + 1) f_j + R \sum_{j=d_0}^{d-1} \varphi^{-\alpha}(n_f, j + 1) f_j]]
\]
Note that \( \varphi(n_0, n_0) = \varphi(n_0)\varphi^{-1}(n_0) = I \) If \( D^\alpha \) is non singular for each \( \alpha \in [0,1] \), (2.2) determines the unique solution of the boundary value problem. If \( D^\alpha \) is singular then, we can only determine best least square solution of the three point boundary value problem. In this case using (1.9), we get a system of equations \( D^\alpha C = f \) where

\[
f = D^{\alpha+1} \left[ \alpha - [\sum M \sum_{j=d_0}^{d-1} \varphi(n_0, j + 1) f_j + N \sum_{j=d_0}^{d-1} \varphi(n_n, j + 1) f_j + R \sum_{j=d_0}^{d-1} \varphi(n_n, j + 1) f_j] \right]
\]

We solve the system of equation (2.3) by using modified QR-algorithm.

3. The Least squares problem:

The least squares (1.5) problem is one of the central problems in numerical linear algebra. Suppose we have a system of equations of the form \( D^\alpha C = f \)

Where \( D^\alpha \in R^{m \times n}, \text{and } m > n \) meaning \( R \) is long and thin matrix and \( f \in R^{m \times 1} \). We wish to find \( C \) for any fixed \( \alpha \in [0,1] \) such that \( D^\alpha C = f \). In general, we can never expect to find a solution \( C \) such that \( D^\alpha C \approx f \).

Formally (Ls) problem can be defined as

\[
\arg \min_C \| D^\alpha C - f \|_2
\]

Let Q be an orthogonal matrix and \( Q^\alpha \in R^{m \times m} \). For each \( \alpha \in [0,1] \) then Q does not change the norm of a vector. If we rotate or reflect a vector, then the vectors length won’t change. Consider why

\[
\| Q^\alpha y \|_2^2 = (Q^\alpha y)^T (Q^\alpha y) = y^T (Q^\alpha)^T (Q^\alpha y) = y^T y = \| y \|_2^2
\]

With this idea is min , consider now an orthogonal matrix can be used for an LS problem.

\[
\begin{align*}
&= \min_C \| D^\alpha C - f \|_2 \\
&= \min_C \| (Q^\alpha)^T (D^\alpha C - f) \|_2 \\
&= \min_C \| (Q^\alpha)^T (QR^\alpha C - f) \|_2 \\
&= \min_C \| (R^\alpha C - QT f) \|_2
\end{align*}
\]

Our goal is to find a Q such that \( Q^\alpha, D^\alpha = Q^\alpha . R^\alpha \) where \( R^\alpha \) is upper triangular for each \( \alpha \in [0,1] \). QR factorization for solving least square problems

In fact QR - decomposition exists for any matrix. Given a matrix our goal is to find two matrices \( Q^\alpha, R^\alpha \) such that Q is orthogonal and \( R^\alpha \) is upper triangular. Here \( D^\alpha \) is a \( m \times p \) matrix and hence

\[
D^\alpha = Q^\alpha \begin{bmatrix} R^\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} Q^\alpha_1 \\ Q^\alpha_2 \end{bmatrix} \begin{bmatrix} R^\alpha \\ 0 \end{bmatrix}
\]

Note that the matrix \( R^\alpha \) will be always square say \( p \times p \)

Consider \( D^\alpha C = f \) \hspace{1cm} (3.2)

If \( D^\alpha \) is an \( m \times p \) matrix with columns linearly independent then

\[
(D^{\alpha T} D^\alpha) C = D^\alpha f \text{ for each } \alpha \in [0,1]
\]

Now \( (D^{\alpha T} D^\alpha) \) is a square matrix of order \( p \) and hence \( C = (D^{\alpha T} D^\alpha)^{-1} D^\alpha f \)
Thus, using the (QR) decomposition yields a better least square estimate than the normal equations in terms of solution quality. In case $D^\alpha$ is a $m \times p$ matrix with rows of $D^\alpha$ are linearly independent, then the transformation $C = (D^\alpha)^T$ gives

$$\left( D^\alpha D^\alpha^T \right) y = D^\alpha f$$
$$y = \left( D^\alpha D^\alpha^T \right)^{-1} D^\alpha f$$

Since $C = D^\alpha^T y$, We have $y = D^\alpha^T \left( D^\alpha D^\alpha^T \right)^{-1} D^\alpha f$ is the unique solution.

**Rank-Deficient Least-square problems:**

When $D^\alpha$ is a square matrix of order $p \times p$, we use least squares algorithm under the assumption $D^\alpha$ is not of full rank. If it is of full rank then the solution of $D^\alpha C = f$ can be determined uniquely. We can use suitable choices the first one is SVD(singular value decomposition) or its cheaper approximations $Q^\alpha R^\alpha$ with column pivoting. If matrix $D^\alpha$ for each $\alpha \in [0,1]$ is rank deficient, then it is no longer the case that space spanned by the columns of $* Q^\alpha *$ is the same space spanned by columns of $* A *$ i.e.,

$$\text{span} \ D_1^\alpha, D_2^\alpha, \ldots, \ D_p^\alpha \cong \text{span} \ q_1^\alpha, q_2^\alpha, \ldots, q_p^\alpha$$

**$Q^\alpha R^\alpha$ application:** The generalized minimum residual (GMRES) algorithm will be presented for solving very large, sparse linear systems of equations by using $Q^\alpha R^\alpha$ decomposition. This decomposition is well known and was in-fact proposed by Saad and Schultz in 1986[]. We make use of this method for developing $Q^\alpha R^\alpha$ algorithm to solve our problems in boundary value problems in boundary value problem. Let $D(\alpha) := D$ and $Q(\alpha): Q$ for any $\alpha \in [0,1]$

Def: Arnold i_single_iter($D, Q, K$):

$$Q = D.\text{dot}(Q[:, k])$$
$$h = m.p.\text{zeros}(K+)$$
$$\text{For } i \text{ in } (k+1):$$
$$h(i)q.T.\text{dot}(Qf[:, i])$$
$$q = h[i] * Q[:, i]$$
$$h(k + 1) = \text{np.linalg.norm}(q^\alpha)$$
$$q /= h[k + 1]$$

Return h,q

Def gmres($DC, x, \text{max_iters}$):

EPSILON =

n . _ = D. Shape

assert ($D^\alpha, \text{shape}[\theta] = D^\alpha.\text{shape}[:]$)

$$r = f - D^\alpha . \text{dot}(C^\alpha)$$
\[ q^a = m.p.zeros((m, \text{max}_\text{iters}) \]
\[ Q[:, \theta] = q.squeeze() \]
\[ beta = m.p.linalg.norm(r) \]
\[ C i = n.p.zeros((n,1)) \]
\[ C i[\theta] =: \neq e - 1 \text{standard basis vector,} \]
\[ H = mp.zeros((n + 1, n)) \]
\[ F = mp(zeros(\text{max}_\text{iters}, n, n)) \]

for i in range (\text{max}_\text{iters}):

\[ F(i) = mp.eye(n) \]

for k in range (\text{max}_\text{iters} - 1):

\[ H[:, k + 2, k], Q^a[:, k + 1] = \text{arhold} i - \text{single}_\text{iter}(D, Q, k) \]

\# don't need to this for \( \emptyset, \ldots, m \) since completed previously

\[ c, s = \text{given coeffs}(H[k, k], H[k + 1, k]) \]
\[ F[k, k, k] = c \]
\[ F[k, k, k + 1] = s \]
\[ F[k, k + 1, k] = -s \]
\[ F[k, k + 1, k + 1] = c \]

\# apply the rotation to both of these
\[ H[:, k + 2, k] = F[k: k + 2: k + 2].dot[H[:, k + 2, k]] \]
\[ ci = F[k].dot(X_i) \]

If \( beta = m.p.linalg.norm(Ci[k + 1]) < \text{epsilon} \)

STOP

\# when terminates, solve the least square problem

\# y must be (k,1)
\[ y; -,-,- = nplinalg.lstsq(H[:, k + 1], \]

\# 0 \( k \) will have dimension \( (m, k)Ci[:, k + 1] \)
\[ C - k = C + Q[:, k + 1].dot(y) \]

return \( C - k \)

Def given_coeffs(a,b)
\[ c = a | np.sqrt(a^2 + b^2) \]
\[ s = b | np.sqrt(a^2 + b^2) \]

```
return c, s
```

Def Arnold i(D, f, k):

```
n = D.shape[0]
H = mp.zeros(k,k)
Q = mp.zeros(m,k)

# normalize the input vector
# use it as the first krylov vector
Q[:,0] = b | np.linalg.norm(f)

for j in range (k - 1):
    Q[:,j + 1] = a.dot(Q[:,j])

for i in range (j):
    H[i,j] = Q[:,j + 1].dot(Q[:,j])

Q[:,j + 1] = Q[:,j + 1] - H[i,j] * Q[:,1]

H[j + 1,j] = np.linalg.norm(Q[:,j + 1])
Q[:,j + 1] = H[j + 1,j]

return Q, H
```

References:


