www.ijecs.in
*International Journal Of Engineering And Computer Science*
*Volume 14 Issue 11 November 2025, Page No. 27945-27957*
*ISSN: 2319-7242 DOI: 10.18535/ijecs/v14i11.5313*

# A Numerical Solution of Second-Order Linear Boundary Value Problems Using Natural Cubic Splines

**Matam Santoshi Kumari**[1]

[1]Department of Mathematics
Chaitanya Bharathi Institute of Technology
Hyderabad,Telangana-500075, India.

**Abstract:**
In this paper, a natural cubic spline method (NCS) has been developed to solve two-point boundary vale problems (BVPS) of second order differential equation. The solution of the BVPs is initially approximated by cubic splines and derived a recurrence relation with natural spline constraints. Replacing this recurrence relation and respective cubic spline approximation in BVPs, obtained a tri-diagonal system of equation. An efficient Thomas algorithm is used to find the solution and represents the results graphically. The famous differential equations from Bessel's equation, Lane-Emden equation, porous catalyst pellet has been considered to check the developed NCS method. Table values for various step sizes are computed in order to verify the developed method's accuracy. The outcomes are contrasted with those obtained using the shooting technique and spectral methods as well as with those found in the literature.

**Keywords:** Natural Cubic Spline, boundary value problem, tri-diagonal system, Thomas Algorithm

## 1. Introduction

"A boundary value problem of the form:

$$y''(x) + p(x)y' + q(x)y = r(x), \;\; a \le x \le b \tag{1.1}$$

subject to the boundary conditions

$$\alpha_0 y(a) + \beta_0 y'(a) = \gamma_0 \tag{1.2}$$

$$\alpha_1 y(b) + \beta_1 y'(b) = \gamma_1 \tag{1.3}$$

called as two-point boundary value problem". This type of boundary value problems can be obtained from various engineering and industrial applications. A two-point BVP specifies the solution at two places along a region's boundaries. Numerous numerical methods have been created to solve two-point BVPs.

An equation of the form $y''(x) + p(x)y'(x) + q(x)y(x) = r(x), \;\; 0 \le x \le 1$ is singular if atleast one of the functions $p(x), q(x)$ and $r(x)$ are not defined at $x = 0$.

Numerous engineering and industrial applications can yield this kind of boundary value problem [6,7]. The solution is specified at two points along the boundaries of a region by a two-point BVP [4]. Two-point BVPs can be solved numerically using a variety of techniques [6].

Numerous fields of science and engineering, including gas dynamics, nuclear physics, atomic structures, and chemical processes, are affected by singular BVP. Most of the time, it is challenging to find an analytical solution to these problems, hence one looks to numerical approaches to discover solutions. Numerous real-world issues involving mathematical modelling involve singular two-point BVPs. Researchers have paid a lot of attention to singular BVPs in ODEs. Due to the singularity behaviour that happens at a point, the numerical solution of SBVP has always been a complex and challenging endeavour. . They arise in the study of conduction of heat in a spherical shell, Unsteady fluid flow through a micro tube, Stability of a flow through a tube, Convection in a fluid sphere [3] etc.

cubic splines to solve a two-point BVP, as done by Albasiny and Hoskins [4], leads to the resolution of a three-term recurrence relationship. When f(x) is constant, a special case in which the approximation applies, it relates to a finite-difference representation.

In physics and engineering, singular two-point boundary value problems are common, but it is frequently difficult or impossible to find analytical solutions for them. As a result, many numerical techniques have been created, such as spline-based methods, optimization-based techniques, and classical finite differences [4,8]. For example, Zhang et al. (2023) suggested a modified cubic B-spline approach to effectively handle singularities [8], while cubic splines have been successfully used to approximate solutions of two-point BVPs [4]. Furthermore, singular two-point BVPs have been successfully solved using metaheuristic optimization techniques like the continuous genetic algorithm, offering a different approach for challenging issues [9]. Expanding on these concepts, natural cubic spline techniques have recently been used to solve hyperbolic and parabolic equations, showing increased accuracy for physical and engineering applications [6,7].

In this paper we considered Natural cubic spline in solving second order singular boundary value problems by considering examples with different types of boundary conditions.

## 2 Natural Cubic Spline Method For Linear Ode

The solution y(x) of eqn. (1.1) can be approximated by the cubic polynomial

$$S(x)=\frac{1}{6h}\left(M_{i-1}(x_i-x)^3+M_i\left(x-x_{i-1}\right)^3\right)+\left(y_{i-1}-\frac{h^2}{6}M_{i-1}\right)\left(\frac{x_i-x}{h}\right)+\left(y_i-\frac{h^2}{6}M_i\right)\left(\frac{x-x_{i-1}}{h}\right) \qquad (2.1)$$

where $M_i=S''(x_i)$ and $y_i=y(x_i)$.

The recurrence relation is given by

$$\frac{h}{6}M_{i-1}+\frac{2h}{3}M_i+\frac{h}{6}M_{i+1}=\left(\frac{y_{i+1}-2y_i-y_{i-1}}{h}\right),\ \ i=1,2,\ldots,n-1. \tag{2.2}$$

From (1.1)

$$M_i=r_i-q_iy_i-p_iS_i' \quad (i=0,1,\ldots,n) \tag{2.3}$$

where $y_i' = S_i'$. Replacing $S_i'$ by $S'(x_i^+)$ and $y_i$ by $S_i$ in (2.3) we get

$$\left(1-\frac{h}{3}p_i\right)M_i-\frac{h}{6}p_iM_{i+1}=r_i-q_iy_i-\frac{p_i}{h}\left(y_{i+1}-y_i\right);\quad i=0,1,2,\ldots,n-1. \tag{2.4}$$

Similarly replacing $S_i'$ by $S'(x_i^-)$ and $y_i$ by $S_i$ in (2.3), we get

$$\frac{h}{6}p_iM_{i-1}+\left(1+\frac{h}{3}p_i\right)M_i=r_i-q_iy_i-\frac{p_i}{h}\left(y_i-y_{i-1}\right);\quad i=1,2,\ldots,n. \tag{2.5}$$

Adding (2.4) and (2.5) we obtain

$$\frac{h}{6}p_iM_{i-1}+2M_i-\frac{h}{6}p_iM_{i+1}=2\left(r_i-q_iy_i\right)-\frac{p_i}{h}\left(y_{i+1}-y_{i-1}\right);i=1,2,..,n-1. \tag{2.6}$$

Eliminating $M_i$ from (2.2) and (2.6), we get

$$\begin{aligned}&\left(1+\frac{h}{3}p_i\right)y_{i+1}-2\left(1-\frac{h^2}{3}q_i\right)y_i+\left(1-\frac{h}{3}p_i\right)y_{i-1}\\ &\qquad=\frac{2h^2}{3}r_i+\frac{h^2}{6}\left(1-\frac{h}{3}p_i\right)M_{i-1}+\frac{h^2}{6}\left(1+\frac{h}{3}p_i\right)M_{i+1}.\end{aligned} \tag{2.7}$$

Replacing $i$ by $i-1$ in (2.4) we have

$$\begin{aligned}&\left(1-\frac{h}{3}p_{i-1}\right)M_{i-1}-\frac{h}{6}p_{i-1}M_i=r_{i-1}-q_{i-1}y_{i-1}-\frac{p_{i-1}}{h}(y_i-y_{i-1});\\ &\qquad i=1,2,\ldots,n.\end{aligned} \tag{2.8}$$

Eliminating $M_i$ from (2.5) and (2.8), we get

$$\begin{aligned}&\left[\left(1+\frac{h}{3}p_i\right)\left(1-\frac{h}{3}p_{i-1}\right)+\frac{h^2}{36}p_{i-1}p_i\right]M_{i-1}\\ &=\left(1+\frac{h}{3}p_i\right)(r_{i-1}-q_{i-1}y_{i-1})-\left(1+\frac{h}{3}p_i\right)\left(\frac{p_{i-1}}{h}\right)(y_i-y_{i-1})\\ &\quad+\frac{h}{6}p_{i-1}(r_i-q_iy_i)-\frac{p_ip_{i-1}}{6}(y_i-y_{i-1});\qquad i=1,2,\ldots,n.\end{aligned} \tag{2.9}$$

Consider

$$\left(1+\frac{h}{3}p_i\right)\left(1-\frac{h}{3}p_{i-1}\right)+\frac{h^2}{36}p_{i-1}p_i=a_i. \tag{2.10}$$

Therefore (2.9) becomes

$$a_i M_{i-1} = \left(1+\frac{h}{3}p_i\right)\left[r_{i-1} - q_{i-1}y_{i-1} - \frac{p_{i-1}}{h}(y_i - y_{i-1})\right] + \frac{h}{6}p_{i-1}\left[r_i - q_i y_i - \frac{p_i}{h}(y_i - y_{i-1})\right]; \quad i = 1,2,\ldots,n. \tag{2.11}$$

Replacing $i$ by $i+1$ in (2.5) we have

$$\frac{h}{6}p_{i+1}M_i + \left(1+\frac{h}{3}p_{i+1}\right)M_{i+1} = r_{i+1} - q_{i+1}y_{i+1} - \frac{p_{i+1}}{h}(y_{i+1} - y_i); \quad i = 0,1,2,\ldots,n-1. \tag{2.12}$$

Eliminating $M_i$ from (2.4) and (2.12), we get

$$\left[\left(1+\frac{h}{3}p_{i+1}\right)\left(1-\frac{h}{3}p_i\right) + \frac{h^2}{36}p_i p_{i+1}\right]M_{i+1} = \left(1-\frac{h}{3}p_i\right)(r_{i+1} - q_{i+1}y_{i+1}) + \frac{p_i p_{i+1}}{6}(y_{i+1} - y_i) - \left(1-\frac{h}{3}p_i\right)\left(\frac{p_{i+1}}{h}\right)(y_{i+1} - y_i) - \frac{h}{6}p_{i+1}(r_i - q_i y_i); \quad i = 0,1,2,\ldots,n-1. \tag{2.13}$$

Consider $b_i = a_{i+1}; \; i = 0,1,2,\ldots,n-1.$

Therefore, we have

$$b_i M_{i+1} = \left(1-\frac{h}{3}p_i\right)\left[r_{i+1} - q_{i+1}y_{i+1} - \frac{p_{i+1}}{h}(y_{i+1} - y_i)\right] - \frac{h}{6}p_{i+1}\left[r_i - q_i y_i - \frac{p_i}{h}(y_{i+1} - y_i)\right] \quad i = 0,1,2,\ldots,n.-1. \tag{2.14}$$

Substituting (2.11) and (2.14) in (2.7) and by simplifying, we get

$$A_i y_{i-1} - B_i y_i + C_i y_{i+1} = D_i, \quad i = 1,2,\ldots,n-1 \tag{2.15}$$

where

$$A_i = b_i\left[1 - \frac{h}{2}p_{i-1} + \frac{h^2}{6}q_{i-1}\right],$$

$$B_i = a_i\left(1 + \frac{h}{2}p_{i+1}\right) + b_i\left(1 - \frac{h}{2}p_{i-1}\right) - \frac{2h^2}{3}q_i d_i,$$

$$C_i = a_i\left[1 + \frac{h}{2}p_{i+1} + \frac{h^2}{6}q_{i+1}\right],$$

$$D_i = \frac{h^2}{6}\left(b_i r_{i-1} + 4d_i r_i + a_i r_{i+1}\right),$$

$$a_i = 1 - \frac{h}{3}p_{i-1} + \frac{h}{3}p_i - \frac{h^2}{12}p_i p_{i-1},$$

$$b_i = 1 - \frac{h}{3} p_i + \frac{h}{3} p_{i+1} - \frac{h^2}{12} p_i p_{i+1},$$

$$d_i = 1 - \frac{h^2}{12} p_{i-1} p_{i+1} + \frac{7h}{24}\left( p_{i+1} - p_{i-1} \right),$$

To solve Eq. (2.15) we write

$$y_{i+1} = W_i y_i + T_i \tag{2.16}$$

Hence equation (2.15) becomes

$$A_i y_{i-1} - B_i y_i + C_i W_i y_i + C_i T_i = D_i$$

$$y_i = \frac{D_i - C_i T_i}{C_i W_i - B_i} - \frac{A_i}{C_i W_i - B_i} y_{i-1}$$

$$y_i = \frac{A_i}{B_i - C_i W_i} y_{i-1} - \frac{C_i T_i - D_i}{B_i - C_i W_i} \tag{2.17}$$

From (2.16)

$$y_i = W_{i-1} y_{i-1} + T_{i-1} \tag{2.18}$$

Comparing (2.17) and (2.18)

$$W_{i-1} = \frac{A_i}{B_i - C_i T_i}, \qquad T_{i-1} = \frac{C_i T_i - D_i}{B_i - C_i T_i} \tag{2.19}$$

To solve these recurrence relations for $W_i$ and $T_i$ $(i = n-2, ...., 0)$ we need to know the values of $W_{n-1}$ and $T_{n-1}$

Boundary conditions for $x = nh$, we have

$$\alpha_0 y_n + \beta_0 y_n{}' = K \tag{2.20}$$

(2.20) can be approximated at $x = x_n$ using

$$S\left(x_n^-\right) = \frac{h}{3} M_n + \frac{h}{6} M_{n-1} + \frac{y_n - y_{n-1}}{h} \tag{2.21}$$

where

$$M_n = \frac{1}{b_{n-1}}[(r_n - \frac{h}{3} p_{n-1} r_n - \frac{h}{6} p_n r_{n-1}) - y_n (q_n - \frac{h}{3} p_{n-1} q_n + \frac{p_n}{h} - \frac{1}{2} p_n p_{n-1}) + y_{n-1} (\frac{p_n}{h} - \frac{1}{2} p_{n-1} p_n + \frac{h}{6} p_n q_{n-1})]$$

and

$$M_{n-1} = \frac{1}{a_n}[(r_{n-1} + \frac{h}{3} p_n r_{n-1} + \frac{h}{6} p_{n-1} r_n) - y_{n-1} (q_{n-1} + \frac{h}{3} p_n q_{n-1} - \frac{p_{n-1}}{h} - \frac{1}{2} p_n p_{n-1}) - y_n (\frac{p_{n-1}}{h} + \frac{h}{6} p_{n-1} q_n + \frac{1}{2} p_{n-1} p_n)]$$

where $b_{n-1}$ and $a_n$ defined by (from tridiagonal system)

Substitute $M_{n-1}, M_n$ in (2.21) and then in (2.20), we get

$$\alpha_1 y_n - \beta_1 y_{n-1} = \gamma_1 \tag{2.22}$$

where

$$\alpha_1 = \alpha_0 + \frac{1}{3}\frac{h}{a_n}\beta_0\left(-q_n - \frac{p_n}{h} - \frac{1}{2h}p_{n-1} + \frac{h}{4}p_{n-1}q_n + \frac{1}{4}p_n p_{n-1} + \frac{3a_n}{h^2}\right)$$

$$\beta_1 = \frac{h}{3a_n}\beta_0\left(-\frac{p_n}{h} - \frac{1}{2h}p_{n-1} + \frac{1}{2}q_{n-1} + \frac{1}{4}p_n p_{n-1} + \frac{3a_n}{h^2}\right)$$

$$\gamma_1 = K + \frac{1}{3}\frac{h}{a_n}\beta_0\left(-r_n - \frac{1}{2}r_{n-1} + \frac{h}{4}p_{n-1}r_n\right)$$

From (2.18) for $i = n$

$$y_n = W_{n-1}y_{n-1} + T_{n-1} \tag{2.23}$$

Comparing (2.22) and (2.23)

$$W_{n-1} = \frac{\beta_1}{\alpha_1} \tag{2.24}$$

$$T_{n-1} = \frac{\gamma_1}{\alpha_1} \tag{2.25}$$

Thus $W_i's$ and $T_i's\,(i = n-2, n-3, \ldots, 0)$ are obtained iteratively in the backward sweep by using (2.24) and (2.25) as the initial values for $W_i's$ and $T_i's$ and knowing values of initial conditions $y_0$, solution $y_i's\,(i = 1, 2, \ldots, n)$ can be obtained by forward process by using Eq. (2.16).

To check the accuracy of the NCS method error analysis has been done. The absolute error, $L_2$ and $L_\infty$ norms are given by

Absolute error $= |\, y_{app} - y_{exact}\,|$

$$L_2 \text{ norm} = \sqrt{\sum_{i=1}^{N} |\, y_{app}(x_i) - y_{exact}(x_i)\,|^2}$$

$$L_\infty \text{ norm} = \max_i |\, y_{app}(x_i) - y_{exact}(x_i)\,|$$

**3 Numerical Examples:**

The developed NCS method is applied to the following examples at different boundary conditions.

**Example 3.1**: **Lane Emden equation**

Consider Lane Emden equation is of the form

$$y''(x) + \frac{1}{x}y'(x) + y(x) = x^2 - x^3 - 9x + 4,$$

Subject to

$$y'(0) = 0, \quad y(1) = 0.$$

The exact solution is $y(x) = x^2 - x^3$.

The singular behaviour that occurs at x = 0 gives the main difficulty for solving the Lane Emden equation.

The BVP is of the form

$$y''(x) + \frac{k}{x} y'(x) + l(x) y(x) = m(x), \; 0 < x < 1.$$

which is non-homogeneous singular boundary value problem with singularity at x=0.

First, we reduce the given singular BVP to linear boundary value problems as follows

$$y''(x) + p(x) y' + q(x) y = r(x), \;\; a \le x \le b$$

subject to boundary conditions:

$$y'(a) = \alpha, \; y(b) = \beta$$

where

$$p(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{x}, & x \neq 0 \end{cases}, \; q(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ 1, & x \neq 0 \end{cases} \quad \text{and} \;\; r(x) = \begin{cases} 2, & x = 0 \\ x^2 - x^3 - 9x + 4, & x \neq 0 \end{cases}$$

The solution of example 3.1 is presented in figure 3.1 and also tabulated in table 3.1. The absolute error of NCS method with exact solutions are compared with other numerical methods and tabulated in table 3.2.

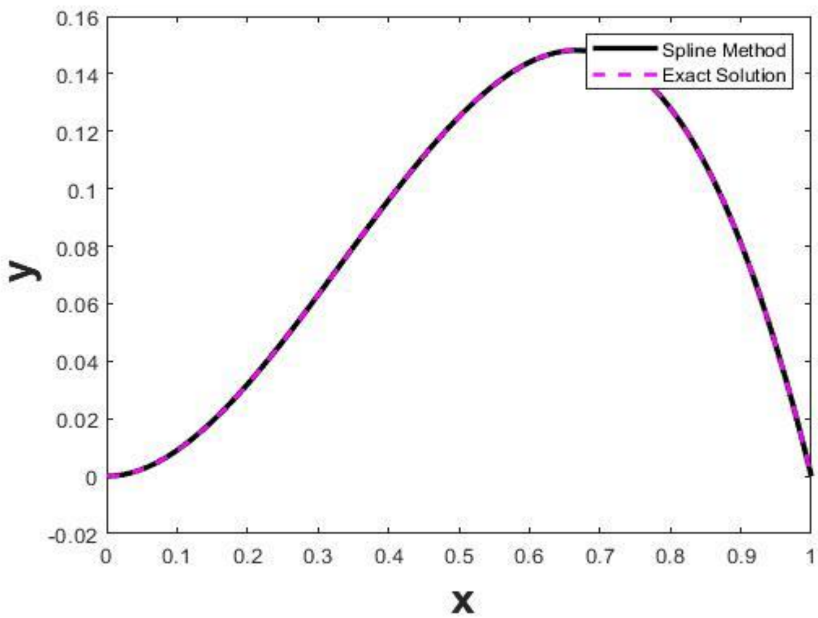


**Figure 3.1:** Solution of the example 3.1 using NCS method and analytical method.

**Table 3.1:** Comparison of NCS with Shooting technique and Spectral Method.

| Step size | Natural Cubic Spline | | Shooting technique | | Spectral Method | |
|---|---|---|---|---|---|---|
| $h$ | Error | Execution time(sec) | Error | Execution time (sec) | Error | Execution time |
| $10^{-1}$ | 2.7061E-16 | 0.035435 | 0.100298 | 0.264670 | 2.7755E-16 | 0.122541 |
| $10^{-2}$ | 2.1806E-14 | 0.038622 | 0.012723 | 0.303117 | 5.9674E-15 | 0.127247 |
| $10^{-3}$ | 1.5250E-12 | 0.051658 | 0.001290 | 0.671894 | 1.6412E-12 | 0.354454 |
| $10^{-4}$ | 1.7026E-10 | 0.17103 | 1.175E-4 | 1.784603 | 1.0397E-10 | 380.452 |

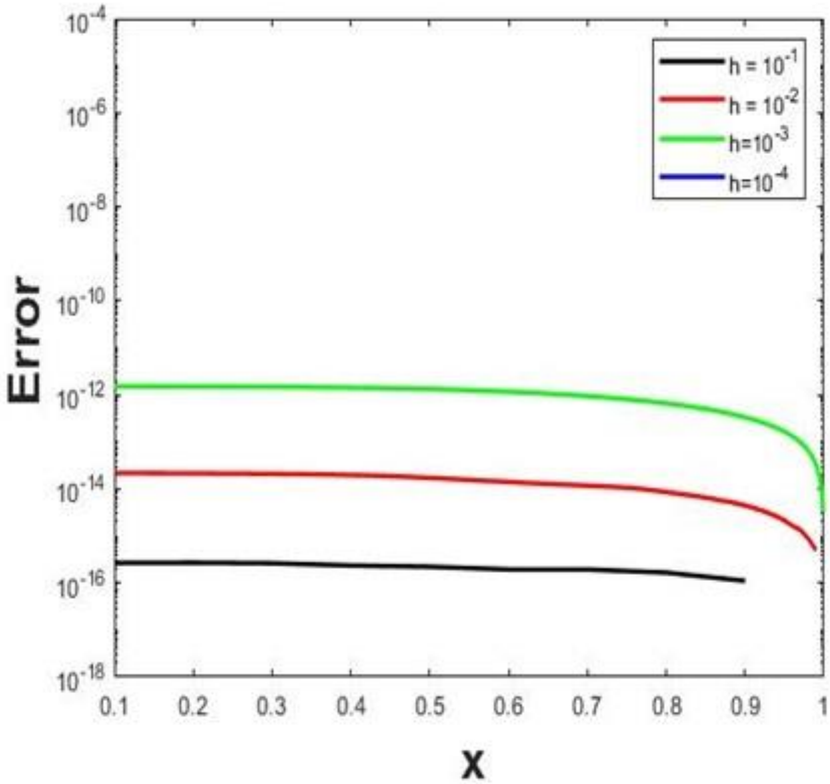


**Figure 3.2:** Absolute error at different step size using NCS method

**Table 3.2:** Comparison of NCS method with published results

| X | NCS | Omar Bu Arqub [5] (Continuous Genetic Algorithm) | Exact |
|---|---|---|---|
| 0.1 | 0.009000000000022 | 0.0089999999973 | 0.009000000000000 |
| 0.2 | 0.032000000000021 | 0.0319999999954 | 0.032000000000000 |
| 0.3 | 0.063000000000021 | 0.0629999999949 | 0.063000000000000 |
| 0.4 | 0.096000000000020 | 0.0959999999950 | 0.096000000000000 |

| | | | |
|---|---|---|---|
| 0.5 | 0.125000000000017 | 0.1249999999952 | 0.125000000000000 |
| 0.6 | 0.144000000000014 | 0.1439999999957 | 0.144000000000000 |
| 0.7 | 0.147000000000012 | 0.1469999999965 | 0.147000000000000 |
| 0.8 | 0.128000000000009 | 0.1279999999976 | 0.128000000000000 |
| 0.9 | 0.081000000000004 | 0.0809999999988 | 0.081000000000000 |

**Example 3.2: Porous Catalyst Pellet**

Consider homogeneous boundary value problem of the form:

$$\frac{d^2C}{dR^2} + \frac{2}{R}\frac{dC}{dR} - (2.236)^2 C = 0, \quad 0 < R < 1$$

with the boundary conditions

$$C'(0) = 0, \quad C(1) = 1.$$

The exact solution is $C(R) = \dfrac{\sinh(2.236R)}{R\sinh(2.236)}$.

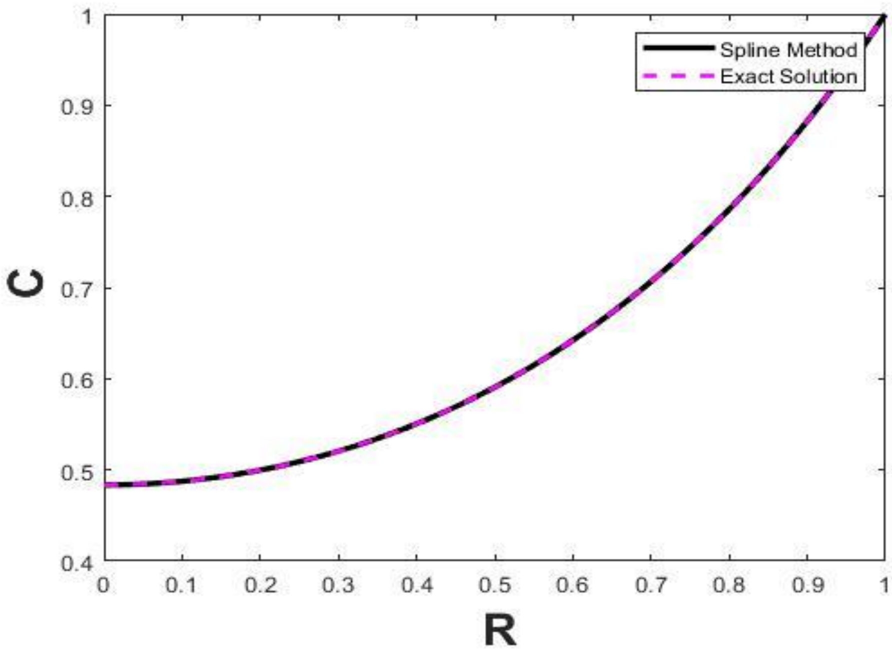


**Figure 3.3:** Comparison of NCS solution with exact solution

**Table 3.3:** Comparison of NCS results with Shooting method and Spectral method at different step sizes

| Step size | Natural Cubic Spline | | Shooting technique | | Spectral Method | |
|---|---|---|---|---|---|---|
| H | Error | Execution time(sec) | Error | Execution time | Error | Execution time |
| 1/10 | 3.27843E-4 | 0.019462 | 0.117296 | 0.233336 | 0.5809 | 0.014854 |
| 1/100 | 3.21786E-6 | 0.020637 | 0.012743 | 0.241956 | 0.4884 | 0.015011 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 1/1000 | 3.21694E-8 | 0.020390 | 0.001273 | 0.478092 | 0.4876 | 0.157218 |
| 1/10000 | 1.2091E-10 | 0.068205 | 1.1588E-4 | 1.055565 | 0.4876 | 125.4423 |

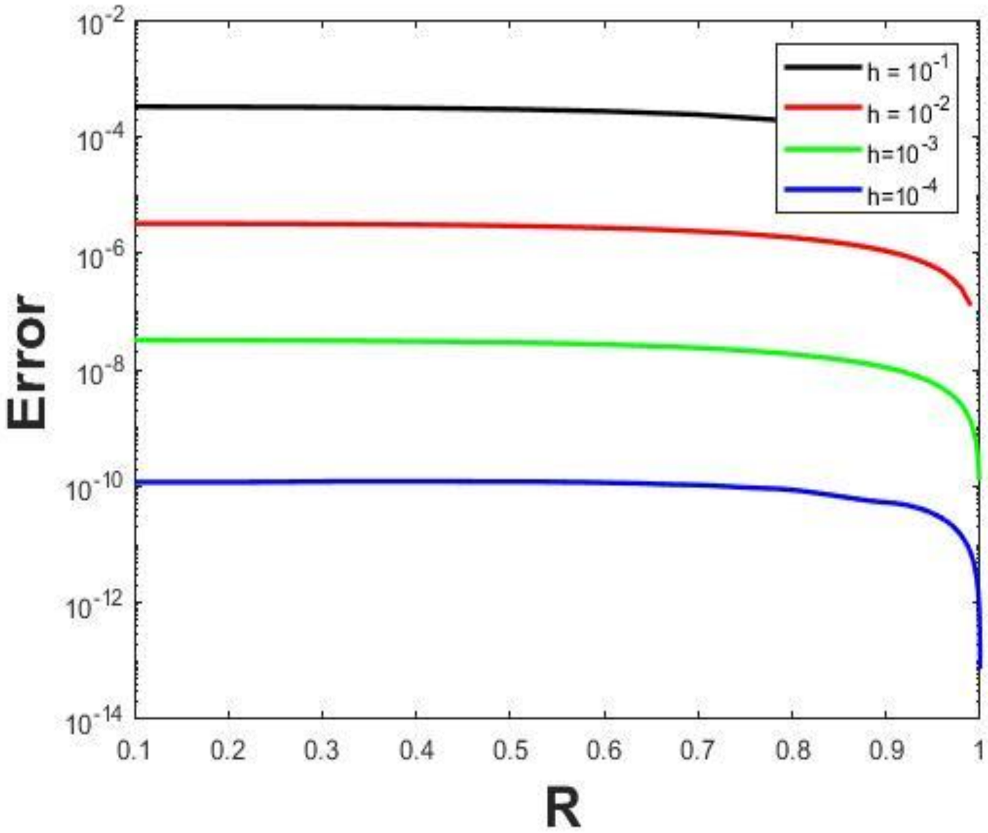


Figure 3.4: Absolute error at different step size using NCS method

**Example 3.3: Bessel's equation:**

Consider another example of singular BVPs, Bessel's equation

$$x^2 y'' + xy' + (x^2 - n^2) y = 0$$

which has singularity at $x = 0$. Bessel's differential equations arise commonly in problems in applied physics and engineering.

In a simplified form, homogeneous Bessel's equation with Dirichlet boundary conditions of the form:

$$y''(x) + \frac{1}{x} y'(x) + y(x) = 0,$$

Subject to:

$$y(0) = \frac{1}{J_0(1)}, \; y(1) = 1.$$

The exact solution is $y(x) = \frac{J_0(x)}{J_0(1)}$.

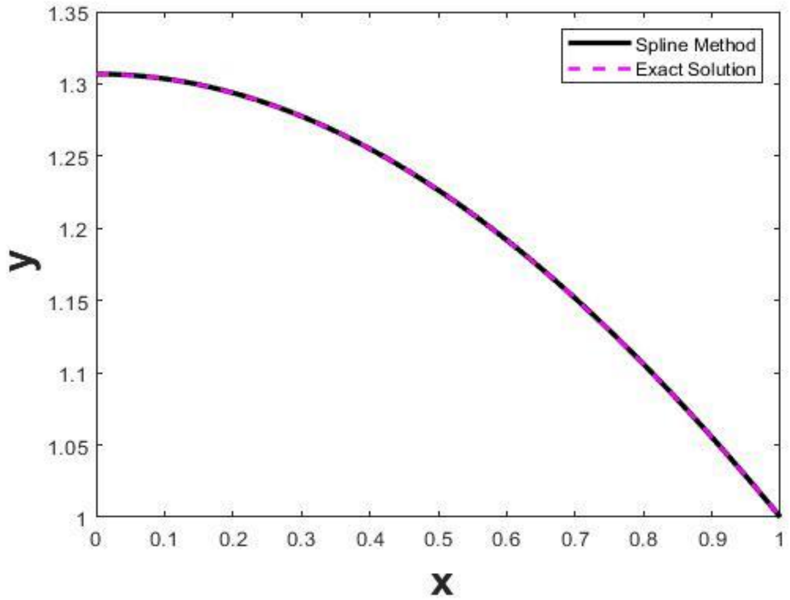


**Figure 3.5:** Comparison of NCS solution with exact solution

**Table 3.4:** Comparison of NCS results with shooting technique and Spectral Method

| Step size | Natural Cubic Spline | | Shooting technique | | Spectral Method | |
|---|---|---|---|---|---|---|
| *h* | Error | Execution time(sec) | Error | Execution time | Error | Execution time |
| 1/10 | 6.47407E-5 | 0.002209 | 0.060091 | 0.519243 | 3.1086E-15 | 0.178608 |
| 1/100 | 7.73654E-7 | 0.004132 | 0.006440 | 0.317347 | 2.4802E-13 | 0.128474 |
| 1/1000 | 8.40803E-9 | 0.009616 | 6.436E-4 | 0.639160 | 1.5804E-11 | 0.315480 |
| 1/10000 | 1.81669E-9 | 1.816699 | 5.504E-5 | 1.833431 | 1.6429E-9 | 153.7733 |

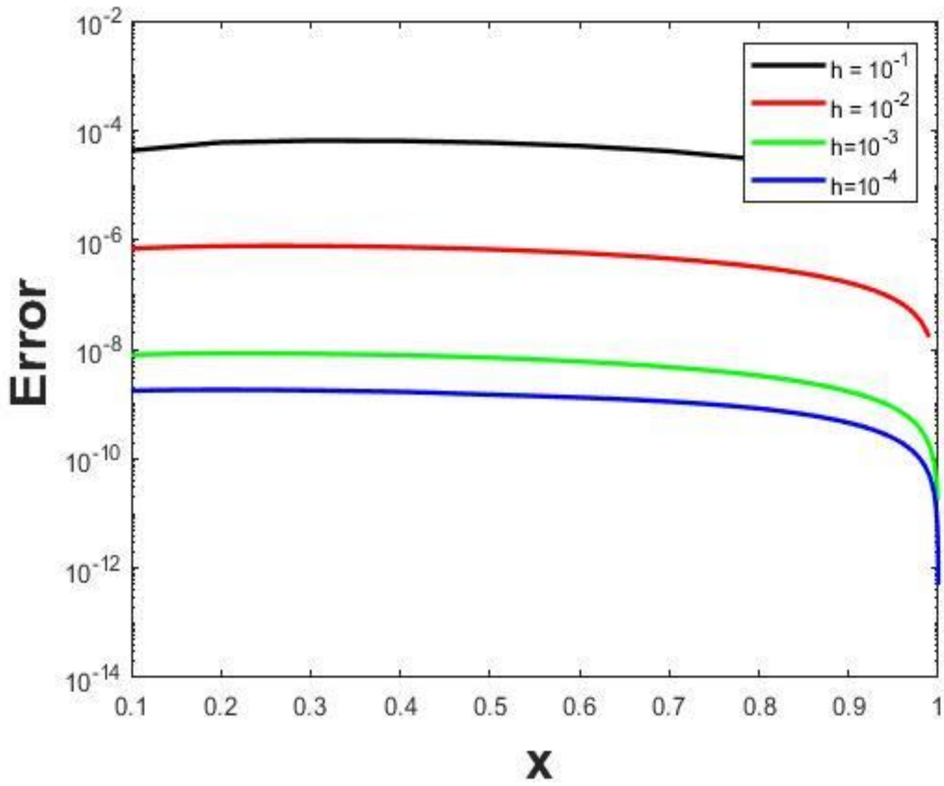


**Figure 3.6**: Absolute error at different step size using NCS method

**4. Conclusion:**

In this paper, a natural cubic spline approach for solving two-point BVPs with cubic splines has been created. A variety of examples are used to illustrate the established NCS approach. Take into account well-known physical issues such the "Lane Emden equation," "diffusion and reaction rate in porous catalyst pellet," and "Bessel's differential equations" are solved using NCS approach. The results of the NCS approach are also contrasted with those of other effective numerical techniques, such as gunshot and spectrum methods. The NCS findings have a strong correlation with previously published research.

**References:**


1. J.H. Ahlberg, E.N. Nilson, J.L. Walsh. The Theory of Splines and their Applications, Academic Press, New York, 1967.
2. H.B. Keller, Numerical Methods for Two Point Boundary Value Problems, Mass Blaisedell, 1968.
3. I.H. Herron, Solving singular boundary value problems for ordinary differential equations, 2013.
4. E.L. Albasiny, W.D. Hoskins, Cubic splines solutions to two-point boundary value problems, Comput.J.12 (1969) 151-153.
5. O.B. Arqub, A.B. Zear, S. Momani, S. Nabil, Solving singular two-point BVPs using Continuous Genetic Algorithm, Abstract and Applied Analysis (2012).
6. M. S. Kumari, H. Y. Shrivalli, B. Mallikarjuna, Natural cubic spline for parabolic equation with constant and variable coefficients, Int. J. Eng. Comput. Sci., 14(7) (2025) 27508–27530. https://doi.org/10.18535/ijecs.v14i07.5188.

7. M. Santoshi Kumari, H. Y. Shrivalli, B. Mallikarjuna, Natural cubic spline for hyperbolic equation with constant coefficients, Commun. Math. Appl., 16 (2025) 127–141.
8. Zhang X., Yang J., Wang Y., Gong D., "A Modified Cubic B-spline Numerical Solution of Singular Two-point Boundary Value Problems", Journal of Systems Science and Mathematical Sciences, 2023. DOI: 10.12341/jssms23576.